Regarding "Arbitrary" Elements...

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Recall that for all integers n, we denote the set of positive integers less than or equal to n as $[n]$.
When writing proofs (particularly inductive ones), it is crucial to keep in mind which items are arbitrary and which items are chosen/constructed by you. In the example question below, we illustrate how mixing this up will lead to an incorrect proof. First, let us define our question.

Example Question:
For $n \in \mathbb{Z}^+$, let T_n denote the set of trees with vertex set $[n]$. Let $T = \bigcup_{n=1}^{\infty} T_n$.
For $t \in T$, t is "happy" if and only if $\exists v \in \mathbb{Z}^+.(\{v, v+1\}$ is an edge of $t)$.
For $n \in \mathbb{Z}^+$, let $P(n)$ denote the predicate "Every tree in T_n is happy".
Prove or disprove: $\forall n \in \mathbb{Z}^+.\big[(n \geq 2) \text{ IMPLIES } P(n)\big]$.

Here's an incorrect student solution. Try to locate the mistake while reading it.

(Incorrect) Student Solution:
Let's prove that $\forall n \in \mathbb{Z}^+.\big[(n \geq 2) \text{ IMPLIES } P(n)\big]$ using induction.
BASE CASE: $P(2)$
The only tree in T_2 is the tree consisting of vertices 1 and 2 and the edge $\{1, 2\}$. As the edge $\{1, 2\}$ is present in this tree, the tree is happy, so $P(2)$ is true.

INDUCTION STEP:
Let $k \in \mathbb{Z}^+$ such that $k \geq 2$. Assume $P(k)$ is true. We will show $P(k+1)$ is true.
Let $t \in T_k$ be given.
Let an arbitrary $w \in [k]$ be given.
Create tree t' by adding the vertex $k+1$ and the edge $\{w, k+1\}$ to t.
As $P(k)$ holds, t must be happy, so there exists $h \in \mathbb{Z}^+$ such that $\{h, h+1\} \in t$.
By construction, t is a subtree of t'. Thus, $\{h, h+1\} \in t'$.
Since t' was constructed from an arbitrary w and t, t' is an arbitrary tree in T_{k+1}.
Thus, $P(k+1)$ holds.

Thus, by induction, $\forall n \in \mathbb{Z}^+.\big[(n \geq 2) \text{ IMPLIES } P(n)\big]$.

Did you find the mistake? If you'd like to find it for yourself, stop reading ahead for now.

The mistake is rather simple: t' is not an *arbitrary* tree from T_{k+1} despite being constructed by an arbitrary t from T_k and connecting the vertex $k+1$ to an arbitrary existing vertex. As the student was the one who made t', t' is a *construction of the student*.

Why does this matter?

Take an arbitrary $k \in \mathbb{Z}^+$ such that $k \geq 2$. For $t \in T_k$ and $w \in [k]$, let $c(t, w)$ denote the tree formed by adding vertex $k+1$ and edge $\{w, k+1\}$ to t. Let $C_k = \big\{c(t, w) \big| t \in T_k,\, w \in [k]\big\}$.

We know that $C_k \subseteq T_{k+1}$, but is it necessarily true that $C_k = T_{k+1}$? If $C_k \neq T_{k+1}$, since we only showed that the trees in C_k are happy, there may still be trees that are not happy in T_{k+1}. This would mean that $P(k+1)$ is not true.

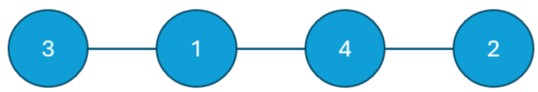
This, in fact, is the case with this proof. Briefly consider the case where $k = 2$. Then, the following tree is an element of T_{k+1}:

When creating the tree t', only one edge is added to t. Thus, the student cannot add both $\{1, 3\}$ and $\{2, 3\}$ to t. This implies that the tree above is not an element of C_k and so $C_k \neq T_{k+1}$.

When proving with induction, it is important to make sure that you consider all instances in each "step". In our example, this is showing that for all $k \in \mathbb{Z}^+$ such that $k \geq 2$, $C_k = T_{k+1}$. Typically, the best approach is to take an arbitrary instance of the "(k+1)-th step" and to find a way to apply the induction hypothesis on a component of the structure. An exception to this rule of thumb is when you prove using structural induction on recursively defined objects.

Still not convinced the student is wrong? Here's a correct solution.

Solution: Observe that the following tree is an element of T_4.



As this tree is not happy, $P(4)$ does not hold, so the statement is false.